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Electrohydrodynamic stability: effects of charge relaxation at the interface of a liquid jet

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The interactions between electrical tractions at the interface of a liquid jet and instability phenomena are studied with emphasis on effects due to interfacial charge relaxation. Charge relaxation causes the oscillatory growth of a perturbation. When viscous effects are small, small fields tend to decrease the growth rate of the axisymmetric mode, up to a point, and precipitate instability of the non-axisymmetric modes. Still larger field strengths increase the growth rates of asymmetric as well as axisymmetric modes. Instabilities characterized by high-frequency oscillations appear to persist even though the charge relaxation phenomena may be quite rapid. When, on the other hand, viscous effects predominate the only unstable disturbance form is the axisymmetric one, although the manner of growth may be oscillatory.

1. Introduction

In electrohydrodynamic phenomena of the sort exemplified by the motion of drops and bubbles in electric fields, the coalescense or disruption of droplets, and the stabilization or disruption of liquid jets, physical and chemical processes at the interface are of paramount importance. These processes are relatively poorly understood and simple models are part of the more comprehensive understanding which is evolving. Simple models involving ohmic conduction and all but ignoring the special nature of interfaces have, moreover, proved useful in exploring a number of phenomena, many of which are discussed in the review by Melcher & Taylor (1969). The subject of this paper is the role of interfacial charge relaxation in the stability of a liquid jet.

Recently Taylor (1969) reported striking behaviour of a liquid jet subjected to a longitudinal electric field. At moderate field strengths the classical Plateau–Rayleigh (varicose) instability is suppressed while at higher field strengths rapid asymmetric motions are produced. High-frequency amplification of axisymmetric and asymmetric disturbances has also been reported by Huebner (1969), who worked with electrically charged jets of water. It has been found that electrical shearing stresses, induced by a deformation of the cylinder, act so as to stabilize the varicose deformations if electrical charges relax instantaneously (Saville 1970). The theory, although consistent with many of the observations, is limited to axisymmetric motions. Furthermore, simply extending it to three dimensions would not lead to a prediction of an oscillatory instability; the disturbance would still be expected to grow or decay exponentially or oscillate without growth. However, oscillatory character might be brought on by electrical effects on a time scale different from those for fluid motion, yet coupled to the motion through the electrical stresses, e.g. the relaxation of electrical charges at an interface. Interfacial charge relaxation, for example, causes overstability at the interface between plane layers of fluid subjected to a tangential field (Melcher & Schwarz 1968).

The relaxation of free charges at an interface is only one of several operative mechanisms in the instability phenomena. In Taylor's experiments, for example, accelerative forces due to gravity and the longitudinal electrical field caused a diminution in cross-section along the jet and the charge induced thereby played a significant role. The complicating phenomena are numerous and it has not been possible to construct an all-inclusive theory. Here, particular attention is given to the effects of one process – charge relaxation.

2. Descriptions of the fluid motions and the electric fields

2.1. *Fluid motions*

The system under study is a cylindrical liquid jet moving axially at a speed w_0 in an electric field of strength E_0, also aligned with the jet; gravitational acceleration is ignored. The liquid is taken as isothermal incompressible and Newtonian and its electrical properties are those of an ohmic conductor with a uniform conductivity and dielectric constant. The electric field influences the motion of the jet only through electrical tractions exerted at the interface between the liquid and the surrounding dielectric cavity.

The location of the interface is represented as

$$\xi = a[1+\zeta(\theta, z', t')], \tag{1}$$

with the deformation, ζ, expressed as a Fourier series with time-dependent coefficients. However, since the magnitude of $\zeta(\theta, z', 0)$ is small and the inquiry centres on the initial stages of growth or decay it suffices to consider a single 'mode',

$$\zeta(\theta, z', t') = \mathrm{Re}\{\zeta(t')\exp[\mathrm{i}(m\theta+kz')]\}. \tag{2}$$

The axial wave-number, k, is $2\pi/\lambda$ (λ is the wavelength) and a represents the radius of the undeformed cylinder. Furthermore, the structure of the linearized equations and boundary conditions implies that $\zeta(t')$ will be exponential, viz. $\zeta_0 \exp(\omega' t')$. The stability parameter, ω', generally depends on k and m and, since the initial growth or decay rate is proportional to the real part of ω', a negative real part implies stability and vice versa.

The linearized equation of motion is

$$\frac{\partial}{\partial t'}\mathbf{v}' + w_0\frac{\partial}{\partial z'}\mathbf{v}' = -\frac{1}{\rho}\nabla' p' + \nu\nabla'^2\mathbf{v}'. \tag{3}$$

Upon transforming the variables according to the scheme

$$\left.\begin{aligned} r' \to ar, \quad z' \to a(z+(w_0/a)\,\tau_0\tau), \quad t' \to \tau_0\tau, \\ \mathbf{v}' \to (T/\rho a)^{\frac{1}{2}}\,\mathbf{v}, \quad p' \to (T/a)\,p, \end{aligned}\right\} \tag{4}$$

with $\tau_0 = (\rho a^3/T)^{\frac{1}{2}}$ the equation of motion becomes

$$\frac{\partial}{\partial \tau}\mathbf{v} = -\nabla p + \frac{1}{R}\nabla^2 \mathbf{v}. \tag{5}$$

Throughout the development the interfacial tension is denoted as T, the density as ρ, the kinematic viscosity as ν, and R stands for $(aT/\rho\nu^2)^{\frac{1}{2}}$. Next, to facilitate finding functions which satisfy (5) and the incompressibility condition, we write

$$\mathbf{v} = \{u(r,\theta,z),\ v(r,\theta,z),\ w(r,\theta,z)\}\exp(\omega\tau),$$

and

$$p = p(r,\theta,z)\exp(\omega\tau), \tag{6}$$

with $\omega = \omega'\tau_0 + iw_0\tau_0/a$ and then define $\mathbf{v}_1$ according to the formula

$$\omega\mathbf{v}_1 = \omega\mathbf{v} + \nabla p. \tag{7}$$

Now the equation for $\mathbf{v}_1$ can be uncoupled from that for p and when the dependence on θ and z is the same as (2) the scalar equations are readily solved yielding:

$$\left.\begin{aligned} u(r,\theta,z) &= -\frac{\alpha A}{\omega}I'_m(\alpha r) - \frac{i\alpha B}{\beta}I'_m(\beta r) + \frac{C}{r}I_m(\beta r),\\ v(r,\theta,z) &= -\frac{imA}{\omega r}I_m(\alpha r) + \frac{m\alpha B}{\beta^2 r}I_m(\beta r) + \frac{i\beta C}{m}I'_m(\beta r),\\ w(r,\theta,z) &= -\frac{i\alpha A}{\omega}I_m(\alpha r) + BI_m(\beta r),\\ \text{and}\quad p(r,\theta,z) &= p_0 + AI_m(\alpha r), \end{aligned}\right\} \tag{8}$$

where $\beta^2 = \alpha^2 + \omega R$ and $\alpha = ka$. The constants A, B and C are to be determined from kinematic and dynamic conditions at the interface. Here the exponential $\exp[i(m\theta + \alpha z)]$ has been consistently omitted. The hydrodynamic stresses will be calculated from these expressions and used to define an equation for ω from which the stability of the system will be inferred but before this characteristic equation can be set up it will be necessary to determine the electrical stresses at the interface.

2.2. *Electric fields and forces*

A number of simplifications of Maxwell's equations are appropriate to the description of electrohydrodynamic phenomena. The fluids under considerations are poor conductors compared with fluids such as mercury or other liquid metals and so induced magnetic effects are small. Thus Maxwell's equations in differential form simplify to (Sommerfeld 1964):

$$\left.\begin{aligned} \nabla' \times \mathbf{E} &= 0,\\ \nabla' . \mathbf{D} &= 4\pi q,\\ \text{and}\quad \partial q/\partial t' + \nabla' . \mathbf{J} &= 0. \end{aligned}\right\} \tag{9}$$

Here the electric field strength is denoted by $\mathbf{E}$, the dielectric displacement is $\mathbf{D}$, the bulk free charge density is q, and the electric current density is $\mathbf{J}$.

The constitutive relations selected to represent the electrical phenomena are

$$\mathbf{D} = K\mathbf{E}, \quad \text{and} \quad \mathbf{J} = \sigma\mathbf{E} + q\mathbf{v}'. \tag{10}$$

The electrical properties of the liquid are the dielectric constant, K, and the electrical conductivity, σ. A characteristic relaxation time for bulk free charges is K/σ, i.e. $q \propto \exp(-\sigma t'/K)$; so if attention centres on regions far from the sources of free electric charge this charge density can be negligibly small. Further information and references to bulk charge relaxation, as well as the constitutive relations, can be found in the survey by Melcher & Taylor (1969). The potential function for the region inside as well as outside the deformed interface can now be found by solving Laplace's equation and considering an interface deformed according to (2) leads to the potential function

$$\left.\begin{aligned} &\hat{\Phi}(r',\theta,z',t') = -E_0 z' + a\hat{A} I_m(kr') \exp[i(m\theta + kz') + \omega' t'] \\ &\text{in the interior and} \\ &\Phi(r',\theta,z',t') = -E_0 z' + aAK_m(kr') \exp[i(m\theta + kz') + \omega' t'] \end{aligned}\right\} \tag{11}$$

in the exterior region. Hereafter carets will be used to distinguish things pertinent to the interior region. The electric field strengths at the interface are composed of the basic field, $(0, 0, E_0)$, plus perturbations $\hat{\mathbf{E}}_1$ and $\mathbf{E}_1$ and these perturbations are

$$\left.\begin{aligned} &\hat{\mathbf{E}}_1 = \{-i\alpha\zeta_0 E_0 - \hat{A}\alpha I'_m(\alpha),\ -im\hat{A} I_m(\alpha),\ -i\alpha\hat{A} I_m(\alpha)\} \\ \text{and} \quad &\mathbf{E}_1 = \{-i\alpha\zeta_0 E_0 - A\alpha K'_m(\alpha),\ -imAK_m(\alpha),\ -i\alpha AK_m(\alpha)\}, \end{aligned}\right\} \tag{12}$$

omitting the exponential factor $\exp[i(m\theta + kz') + \omega' t']$ in each instance. The components of $\hat{\mathbf{E}}_1$ and $\mathbf{E}_1$ are in the directions of a normal to the interface and the tangent directions with senses of increasing θ and z', respectively, and these expressions contain the constants A and $\hat{A}$ which must be determined from the electrical boundary conditions.

The boundary conditions arising from physical arguments are (see Sommerfeld 1964, or Melcher & Taylor 1969): (i) continuity of the tangential components of $\mathbf{E}$, (ii) conservation of electric charge,

$$\partial Q/\partial t' + \langle \sigma E_n \rangle + \nabla_s \cdot (Q\mathbf{v}') = 0. \tag{13}$$

The jump in σE_n across the interface is $\langle \sigma E_n \rangle$, i.e. $\sigma E_n - \hat{\sigma}\hat{E}_n$, the surface charge Q is $\langle KE_n \rangle$ while $\nabla_s \cdot$ denotes the surface divergence. The first boundary condition implies

$$\hat{A} I_m(\alpha) = AK_m(\alpha). \tag{14}$$

Since the interfacial charge density is zero in the undisturbed state the surface divergence term of (13) reduces to $w_0 \partial Q/\partial z'$ after linearization, and, in the convected co-ordinate system specified by (4), (13) becomes

$$\partial Q/\partial \tau + \tau_0 \langle \sigma E_n \rangle = 0. \tag{15}$$

From this equation it follows that

$$\hat{A} = -i\zeta E_0 K_m(\alpha) \Delta(\alpha, \omega), \tag{16}$$

where

$$\Delta(\alpha,\omega) = \frac{(1 - K/\hat{K})\,\omega + \hat{\sigma}\tau_0/\hat{K}}{[I'_m(\alpha)\,K_m(\alpha) - (K/\hat{K})\,I_m(\alpha)\,K'_m(\alpha)]\,\omega + (\hat{\sigma}\tau_0/\hat{K})\,I'_m(\alpha)\,K_m(\alpha)}.$$

The parameter $\hat{\sigma}\tau_0/\hat{K}$ specifies the importance of interfacial charge relaxation and if it is quite large then the interface behaves similar to that between a perfect conductor and a dielectric, i.e. $\hat{\sigma}\hat{E}_n = 0$. Conversely, the behaviour of the interface approximates that between two perfect dielectrics, i.e. $\langle KE_n \rangle = 0$, if $\hat{\sigma}\tau_0/\hat{K}$ is small.

Proceeding to calculate the electric stress tensor (Landau & Lifshitz 1960) one finds the dimensionless components at the interface to be

$$\left.\begin{aligned}\hat{\tau}_{nn}^{(e)} &= -(a\hat{K}E_0^2/8\pi T)\{1-2\alpha\zeta I_m(\alpha)\,K_m(\alpha)\,\Delta(\alpha,\omega)\},\\ \hat{\tau}_{nt}^{(e)} &= -(a\hat{K}E_0^2/4\pi T)\,i\alpha\zeta[1-I_m'(\alpha)\,K_m(\alpha)\,\Delta(\alpha,\omega)],\\ \tau_{nn}^{(e)} &= (K/\hat{K})\,\hat{\tau}_{nn}^{(e)},\\ \tau_{nt}^{(e)} &= -(aKE_0^2/4\pi T)\,i\alpha\zeta[1-I_m(\alpha)\,K_m'(\alpha)\,\Delta(\alpha,\omega)].\end{aligned}\right\} \quad (17)$$

Here the scale for stress is T/a and ζ represents the exponential

$$\zeta_0 \exp\left[i(m\theta+\alpha z)+\omega\tau\right].$$

The tangential component whose sense is in the direction of increasing θ is $O(\zeta_0^2)$ and is therefore neglected.

2.3. *Boundary conditions*

The kinematic and dynamic conditions which must be enforced at the interface between the jet and the dielectric cavity are:

(i) the kinematic condition, $\quad d\zeta/d\tau = u;$ (18)

(ii) continuity of the normal stress,

$$-p+\frac{2}{R}\frac{\partial u}{\partial r}+\hat{\tau}_{nn}^{(e)} = -p_0-1+(1-m^2-\alpha^2)\,\zeta+\tau_{nn}^{(e)}; \quad (19)$$

(iii) continuity of the axially directed shearing stress,

$$\frac{1}{R}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)+\hat{\tau}_{nt}^{(e)} = \tau_{nt}^{(e)}; \quad (20)$$

(iv) continuity of the azimuthally directed shearing stress,

$$r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{1}{r}\frac{\partial u}{\partial \theta} = 0. \quad (21)$$

These four conditions allow $\zeta(\theta, z, r)$ to be specified in terms of A, B and C and lead to the condition which must obtain so as to preclude the trivial solution. Thus ω must be a root of

$$\det D_1 + \frac{Rk_2(\alpha,\omega)}{2\omega}\det D_2 + \frac{R\Omega(\alpha,\omega)}{\alpha^3\omega}\det D_3 = 0. \quad (22)$$

The elements of the square matrices D_1, D_2 and D_3 are

(i) D_1,

$$\left.\begin{aligned}&d_{11} = 1-\alpha I_m'(\alpha)/I_m(\alpha), \quad d_{12} = -1+\beta I_m'(\beta)/I_m(\beta),\\ &d_{13} = 1+\beta^2/2m^2-\beta I_m'(\beta)/m^2 I_m(\beta),\\ &d_{21} = \alpha I_m'(\alpha)/I_m(\alpha), \quad d_{22} = -(\beta^2+\alpha^2)\,\beta I_m'(\beta)/2\alpha^2 I_m(\beta),\\ &d_{23} = -\tfrac{1}{2},\\ &d_{31} = (\beta^2-\alpha^2)/2\alpha^2+2I_m''(\alpha)/I_m(\alpha), \quad d_{32} = -2\beta^2 I_m''(\beta)/\alpha^2 I_m(\beta),\\ &d_{33} = [2-2\beta I_m'(\beta)/I_m(\beta)]/\alpha^2.\end{aligned}\right\} \quad (23)$$

(ii) D_2 is the same as D_1 except for the second row which is

$$d_{21} = \alpha I'_m(\alpha)/I_m(\alpha), \quad d_{22} = -\beta I'_m(\beta)/I_m(\beta), \quad d_{23} = -1. \tag{24}$$

(iii) D_3 is the same as D_1 except for the third row which is

$$d_{31} = -\alpha I'_m(\alpha)/I_m(\alpha), \quad d_{32} = \beta I'_m(\beta)/I_m(\beta), \quad d_{33} = 1. \tag{25}$$

Here
$$\Omega(\alpha, \omega) = \alpha(1-m^2-\alpha^2)+\alpha^2 k_1(\alpha, \omega), \tag{26}$$

$$k_1(\alpha, \omega) = (aKE_0^2/4\pi T)(1-\hat{K}/K)\, I_m(\alpha)\, K_m(\alpha)\, \Delta(\alpha, \omega), \tag{27}$$

and

$$k_2(\alpha, \omega) = (aKE_0^2/4\pi T)\,[1-\hat{K}/K-\{I_m(\alpha)\,K'_m(\alpha) - (\hat{K}/K)\, I'_m(\alpha)\, K_m(\alpha)\}\Delta(\alpha, \omega)], \tag{28}$$

with $\Delta(\alpha, \omega)$ given in (16).

In (22) effects due to electrical shearing stresses, which are associated with det D_2, and electrical normal stresses, associated with det D_3, are clearly distinguished.† Electrical normal stresses have a stabilizing effect in instances where the relaxation parameter $\hat{K}/\hat{\sigma}\tau_0$ is either very small or very large since then $k_1(\alpha, \omega)$ is negative and so increases in the field strength decrease $\Omega(\alpha, \omega)$. The effect is somewhat more complicated, however, in intermediate situations due to coupling of the normal stresses and charge relaxation and effects due to electrical shearing stresses are even more obscure.

3. Stability of the jet when viscous effects are small

Although it is difficult to draw detailed interferences directly from (22) the situation simplifies considerably when R is either very small or very large, e.g. the value of R would be about 600 for a 1 cm diameter water jet in air. The latter situation does not correspond to the complete neglect of viscous effects, for they are simply confined to a thin region near the interface. In this electrohydrodynamic boundary layer viscous effects balance the electrical shear stresses. The necessary formulas are developed as follows. First

$$\beta^2 = \omega R(1+\alpha^2/\omega R), \tag{29}$$

so ω will be $O(1)$ if β is $O(R^{\frac{1}{2}})$. Effects on the slower time scale, viz. where ω is $O(R^{-1})$ and β is $O(1)$, will not be considered. Thus

$$\beta \sim (\omega R)^{\frac{1}{2}}\left[1+\frac{\alpha^2}{2\omega R}+\ldots\right], \quad \left|\frac{\alpha^2}{\omega R}\right| \to 0. \tag{30}$$

Using the first two terms of the asymptotic expansion for $I_m(\alpha)$ given by Watson (1966) leads to

$$\omega^2 - \Omega(\alpha, \omega)\frac{I'_m(\alpha)}{I_m(\alpha)} + \alpha^2 k_2(\alpha, \omega) = O(\beta^{-1}), \quad |\omega R| \to \infty, \tag{31}$$

† For axisymmetric disturbances ($m = 0$) in the absence of electrical effects, (22) reduces to the result cited in Chandrasekhar (1961).

or, written out in full,

$$\tau_r \omega^3 + f_1(\alpha)\,\omega^2 - \tau_r\left[f_0(\alpha)(1-m^2-\alpha^2) - E\frac{K}{\hat{K}}\left(1-\frac{\hat{K}}{K}\right)^2 \alpha^2 f_1(\alpha)\right]\omega$$

$$-\left[f_0(\alpha)(1-m^2-\alpha^2) - E\frac{\hat{K}}{K}\frac{\alpha^2}{f_1(\alpha)}\right] f_1(\alpha) = 0. \qquad (32)$$

Here

$$\left.\begin{aligned} E &= aKE_0^2/4\pi T, \quad f_0(\alpha) = \alpha I_m'(\alpha)/I_m(\alpha),\\ f_1(\alpha) &= \left[1-\frac{K}{\hat{K}}\frac{I_m(\alpha)\,K_m'(\alpha)}{I_m'(\alpha)\,K_m(\alpha)}\right]^{-1}, \quad \text{and} \quad \tau_r = \hat{K}/\hat{\sigma}\tau_0. \end{aligned}\right\} \qquad (33)$$

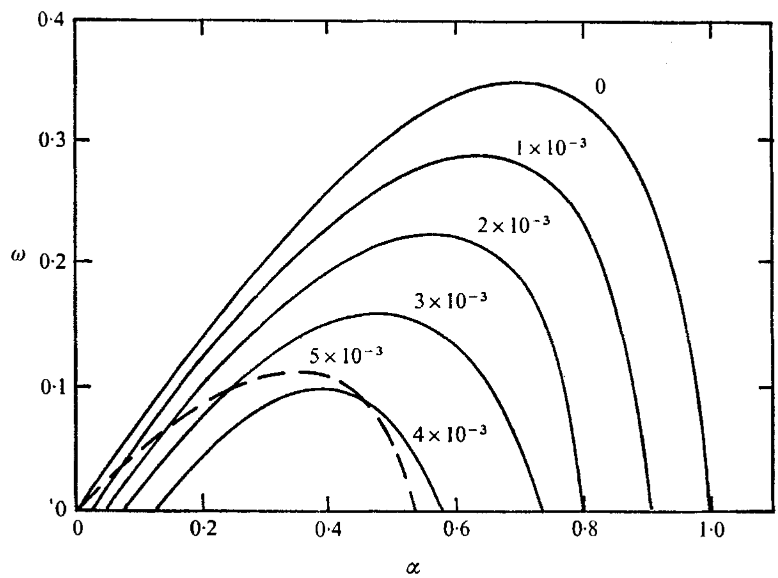

FIGURE 1. The growth rate parameter, ω, for the axisymmetric mode as a function of α for instantaneous charge relaxation; $\hat{K}/K = 78$ and $R \to \infty$. Numbers associated with each curve denote the value of E. The dashed curve shows the ω–α relation for $E = 5 \times 10^{-3}$ with a perfectly insulating fluid.

Two conclusions drawn directly from (32) are: (*a*) $\tau_r \to \infty$. Here the characteristic equation reduces to the form appropriate for perfect dielectrics and no electrical shearing stresses are present. Non-axisymmetric deformations ($m \neq 0$) are always stable; axisymmetric deformations with wave-numbers within the range 0–α_c; $\alpha_c < 1$ are unstable. As E is increased the critical wave-number decreases, although instability persists at small wave-numbers. The formula for the axisymmetric case was first reported by Nayyar & Murty (1960). (*b*) $\tau_r = 0$. This form is appropriate for good conductors and deformations of the type $m \neq 0$ are again stable. Axisymmetric deformations may grow, depending on the magnitudes of E and α, but moderate sized values of E are sufficient to stabilize the jet. For example, within $\hat{K}/K = 78$ stability is indicated when E exceeds 5×10^{-3} (see figure 1); fluids with small dielectric constants require correspondingly larger fields for stabilization. The stabilizing effects due to electrical shear stresses were identified in an earlier study which centred on axisymmetric

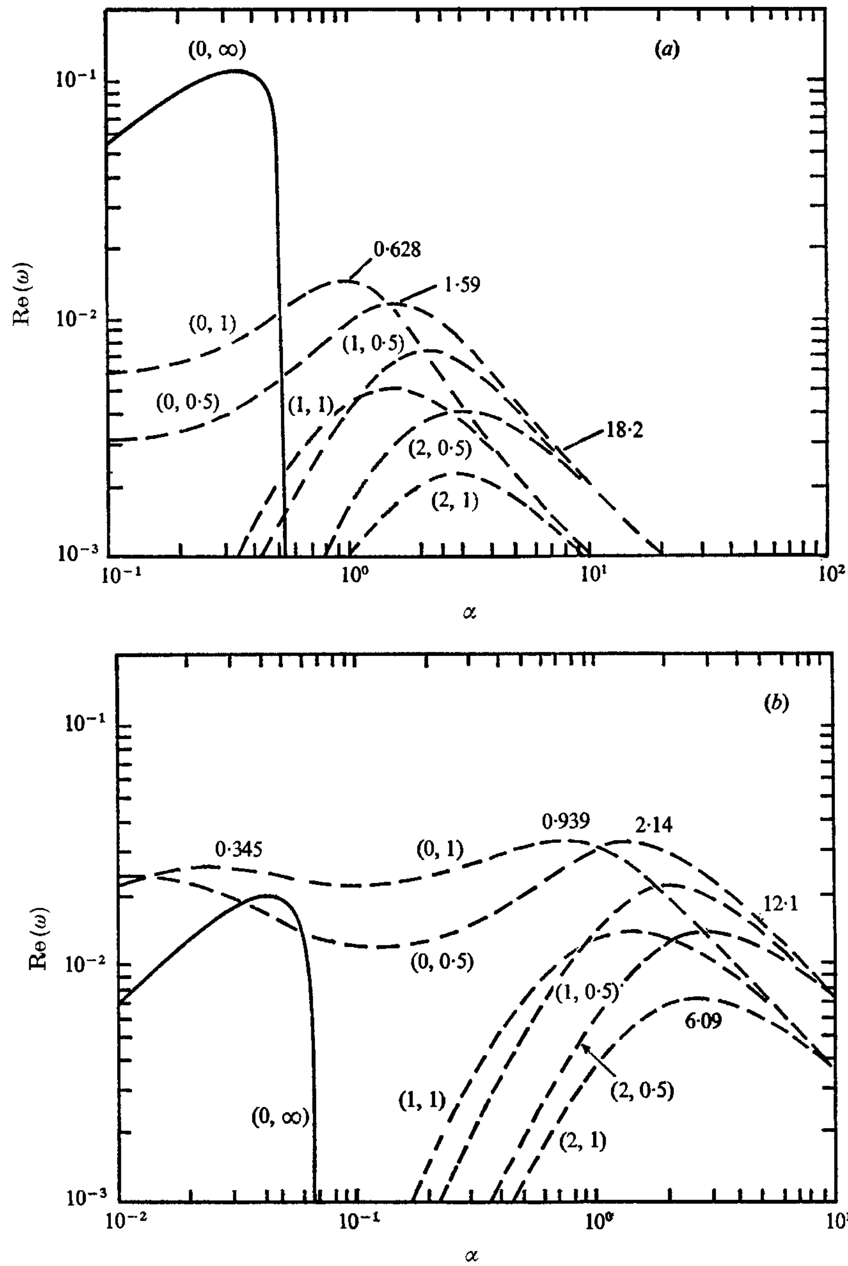

FIGURE 2. The real part of the growth rate parameter, Re (ω), for $R \rightarrow \infty$ as a function of α and the relaxation time τ_r; (*a*) $E = 0{\cdot}005$; (*b*) $E = 0{\cdot}02$. Numbers in parentheses associated with each curve denote the mode and τ_r, respectively. Dashed lines indicate the growth is oscillatory with selected values of Im (ω) indicated.

motions and the effects of the surrounding fluid but excluded charge relaxation effects (Saville 1970).

Next we look for evidence of instability when interfacial charge relaxation is present, i.e. $0 < \tau_r < \infty$, since it is here that electrical stresses could interact with the other stresses to produce oscillatory forms of instability. Equation (32) is cubic in ω so it is a straightforward matter to find the roots, given values for the other variables and parameters; some of the results of numerical calculations are presented as table 1 and on figures 2(*a*) and (*b*). First of all the anticipated electrical interaction should be easy to identify, if it exists at all, in situations where the parameter τ_r has a value near unity. For example, the appropriate time scales for a 1 mm diameter distilled water jet are $\hat{K}/\hat{\sigma} = 7 \times 10^{-4}$ sec and $\tau_0 = 10^{-3}$ sec so that $\tau_r = 0{\cdot}7$. Figures 2(*a*) and (*b*) show results calculated with the field strength parameter, E, equal to 5×10^{-3} and 2×10^{-2}, respectively. In both instances the jet would be stable if interfacial charge distributions could relax instantaneously. Conversely, the axisymmetric mode would be unstable if the relaxation processes were exceptionally slow. It was found that the instability could be oscillatory, i.e. the real part of ω is positive, and unexpectedly, both the asymmetric and axisymmetric types of deformations are unstable. Qualitatively similar behaviour was found for $\hat{K}/K$ less than 78.

The data summarized in table 1 reveal another effect of the longitudinal electric field: the growth rate of the most unstable wavelength of the axisymmetric mode is first decreased then increased as the field strength is raised but the rate of growth of the sinuous mode increases uniformly with increasing field strength.

All these effects are due to electrical relaxation processes; the reasons for their appearance are established by examining equation (32). Recall, first, that when $\tau_r = \infty$ there will always be a range of wave-numbers where instability is possible. Upon writing $\omega = \omega_0 + \delta$, where ω_0 is a root of (32) corresponding to $\tau_r = \infty$ while δ represents a small connexion due to a finite electrical relaxation time, we find

$$\delta \sim -\frac{E(\hat{K}/K)\,\alpha^2[1-f_1^2(\alpha)\,(1-K/\hat{K})^2]}{2\omega_0^2\tau_r}, \quad \tau_r \to \infty. \tag{34}$$

The term in brackets is always positive. Suppose, now, that ω_0^2 is positive, e.g. the disturbance is axisymmetric and unstable with a wave-number in the range 0–α_c. Equation (34) shows that the addition of relaxation processes tends to reduce the rates of growth of these disturbances and the size of the reduction increases with the field strength. At the same time, however, all the stable oscillations ($\omega_0^2 < 0$), are made unstable and their growth rates increase with the field strength.

The results presented here are in qualitative agreement with Taylor's experiments (Taylor 1969) but comparisons are tenuous since, in most of the experiments, the acceleration of the jet in the gravitational field caused the cross-section to vary and hence a surface charge was induced by the electric field. In one of his experiments a continuous thread of water was established between a nozzle and a plate and the potential difference between the plate and the nozzle was decreased. At 4 kV small axisymmetric corrugations were visible; at 3 kV the jet broke up into drops. Although the geometrical configuration of the apparatus made an accurate determination of the field strength parameter, E, difficult the numerical value was estimated as 0·003 for a potential difference of 4 kV. If the liquid were assumed to be a perfect conductor then, according to the present theory, the jet would become stable when E is about 0·005 while if τ_r were as large as 0·05 then the growth rate of the most unstable mode would be reduced by a factor of 28, compared to the zero-field growth rate. This low rate could be sufficient to suppress the appearance of an instability due to the lack of sufficient time for growth and, according to the theory developed here, a much larger field strength would be necessary to increase the growth rate of either the axisymmetric or sinuous instability to a point where it would be comparable in size to, say, the growth rate in the absence of a field. Although Taylor observed a sinuous form of instability at a potential difference of 17 kV with a free jet the instability was absent when the jet was formed round a silk thread which kept the jet cross-section more-or-less uniform.

	$m = 0$			$m = 1$		
E	α	Re (ω)	Im (ω)	α	Re (ω)	Im(ω)
			$\tau_r = 0{\cdot}5$			
0	0·70	0·34	0	stable		
$2{\cdot}5 \times 10^{-3}$	0·53	0·20	0	2·1	0·0039	3·1
$5{\cdot}0 \times 10^{-3}$	1·6	0·012	1·6	2·1	0·0072	3·2
$7{\cdot}5 \times 10^{-3}$	1·6	0·016	1·7	2·1	0·010	3·3
$1{\cdot}0 \times 10^{-2}$	1·6	0·020	1·9	2·1	0·013	3·4
$1{\cdot}5 \times 10^{-2}$	1·5	0·027	1·9	2·1	0·018	3·7
$2{\cdot}0 \times 10^{-2}$	1·5	0·032	2·1	2·1	0·021	3·9
			$\tau_r = 0{\cdot}01$			
5.0×10^{-3}	27	0·025	140	27	0·025	140
$2{\cdot}0 \times 10^{-2}$	27	0·096	143	27	0·096	145

TABLE 1. Characteristics of the most unstable disturbances ($\hat{K}/K = 78$)

Another interesting aspect of the theory is encountered as the relaxation parameter, τ_r, is made smaller and smaller with the field strength parameter fixed but large enough to ensure stability when $\tau_r = 0$. The maxima in the Re $(\omega) - \alpha$ relation now occur at larger and larger wave-numbers and the magnitudes increase somewhat. Thus for $E = 5 \times 10^{-3}$ the wave-number of the most unstable axisymmetric disturbance increases from 1.6 to 27·0 as τ_r decreases from 0·5 to 0·01 (see table 1). It might have been expected that the real part of the complex roots of (32) would be uniformly small, whatever the wave-number, as long as τ_r is small; such is not the case. Some insight into the actual behaviour can be obtained by examining (32) again. First of all when $\alpha \to 0$ or $\alpha \to \infty$ the roots of (32) indicate stability as long as τ_r is positive. Next, with the wave-number fixed, the real root is negative and $O(\tau_r^{-1})$ while the complex roots are:

$$\omega \sim \pm\{f_0(\alpha)(1-m^2-\alpha^2) - E(K/\hat{K})\,\alpha^2 f_1^{-1}(a)\}^{\frac{1}{2}} + \tfrac{1}{2}E\frac{\hat{K}}{K}\alpha^2 f_1^{-2}(\alpha)\left\{1-\left[\frac{f_1(\alpha)(K-\hat{K})}{\hat{K}}\right]^2\right\}\tau_r, \quad (\tau_r \to 0). \quad (35)$$

In the situations under consideration the first term on the right is imaginary while the second term is $O(\tau_r)$, positive, and increases with α. Thus, a shift in the location of the maximum of $\mathrm{Re}(\omega)$ is indicated and, although $\mathrm{Re}(\omega)$ may be made small at a given wave-number by decreasing τ_r, there will always be a maximum in the $\mathrm{Re}(\omega)-\alpha$ relation. Furthermore, the oscillation frequency will be high since $\mathrm{Im}(\omega) \sim \alpha^{\frac{3}{2}}$ as $\alpha \to \infty$.

The further effects of viscosity can be estimated by including the term in (31) which is $O(\beta^{-1})$. In the absence of electrical relaxation phenomena the inclusion of this $O(\beta^{-1})$ term leads to a quintic equation for $\omega^{\frac{1}{2}}$ and if relaxation is included then the equation is of the seventh order. Neither of these situations has been investigated. However, for $\tau_r = 0$ it is easy to show that if the roots of (32) are imaginary, i.e. the motion is stable, then the $O(\beta^{-1})$ effects are always stabilizing. In any event when $\mathrm{Re}(\beta) < 0$ the next term in (31) is $-2\alpha^2 f_0(\alpha)\, k_2(\alpha,\omega)\,\beta^{-1}$ and if $\mathrm{Re}(\beta) > 0$ the sign is reversed. Writing $\omega = \omega_0 + \delta$, where δ is a small correction due to the electrical shearing stresses, leads to

$$\delta = \mp \frac{E\alpha^2}{\omega_0^{\frac{3}{2}} R^{\frac{1}{2}} I_m(\alpha)\, K_m(\alpha)} \tag{36}$$

for $\tau_r = 0$; the negative sign applies for $\mathrm{Re}(\beta) < 0$. Thus for oscillatory modes ($\omega_0 = \pm i\gamma,\ \gamma > 0$) δ is always negative and so the electrical shear stresses have a stabilizing effect. The stabilizing effects of viscosity are likely to be particularly important in cases where the relaxation time is short for there the wavelength is small and oscillation frequency high.

The oscillatory nature of the instability phenomena found here is similar to that identified by Melcher & Schwarz (1968) in their study of the effects of charge relaxation on the stability of a horizontal planar interface in the presence of gravity and a parallel electric field (Rayleigh–Taylor problem). Of course there are some fundamental differences in the situations themselves, so that considerable care must be exercised in attempts at generalization.

For example, interfacial tension has a stabilizing effect on surface waves in an unstable configuration with a plane interface, i.e. surface tension limits the range of unstable wave-numbers, but with a cylindrical interface surface tension may either act so as to cause instability, with axisymmetric deformations, or stability, with non-axisymmetric deformations.

4. Stability of the jet when viscous effects dominate

If the viscous forces predominate then a straightforward expansion of (22) leads to

$$-\omega' \rho v a/T \sim \Omega(\alpha,\omega)\,F(\alpha) + k_2(\alpha,\omega)\,G(\alpha) \quad (|\omega R| \to 0). \tag{37}$$

$F(\alpha)$ and $G(\alpha)$ are, in general, combinations of the determinants of several 3×3 matrices. After writing $\Omega(\alpha,\omega)$ and $k_2(\alpha,\omega)$ in full we have the analogue to (32)

$$\begin{aligned}\bar{\tau}_r\bar{\omega}^2 &+ \{f_1(\alpha) + \bar{\tau}_r F(\alpha)\,[\alpha(1-m^2-\alpha^2) - E(K/\hat{K})\,(1-\hat{K}/K)^2 f_2(\alpha)]\}\,\bar{\omega} \\ &+ \{F(\alpha)\,[f_1(\alpha)\,\alpha(1-m^2-\alpha^2) + E(1-\hat{K}/K)\,f_2(\alpha)] + G(\alpha)\,Ef_3(\alpha)\} = 0,\end{aligned} \tag{38}$$

where $f_1(\alpha)$ is given by (33),

$$\left.\begin{aligned} f_2(\alpha) &= \alpha^2 I_m(\alpha)\, K_m(\alpha) f_3(\alpha), \\ f_3(\alpha) &= [I'_m(\alpha)\, K_m(\alpha) - (K/\hat{K})\, I_m(\alpha)\, K'_m(\alpha)]^{-1} \\ \text{and} \quad \overline{\omega} &= \omega' \rho\nu a/T, \quad \overline{\tau}_r = \hat{K}T/\rho a\nu\hat{\sigma}. \end{aligned}\right\} \quad (39)$$

Numerical studies of (38) uncovered no asymmetric instabilities, so the following remarks will centre on axisymmetric modes. For axisymmetric disturbances $F(\alpha)$ and $G(\alpha)$ are quite simple, viz.

$$\left.\begin{aligned} F(\alpha) &= -(2\alpha)^{-1}[\alpha^2 I_0^2(\alpha)/I_1^2(\alpha) - \alpha^2 - 1]^{-1} \\ \text{and} \quad G(\alpha) &= -\alpha I_0(\alpha)/I_1(\alpha)\,[\alpha I_0(\alpha)/I_1(\alpha) - \alpha I_1(\alpha)/I_0(\alpha) - 1]\, F(\alpha). \end{aligned}\right\} \quad (40)$$

First consider the limiting situation where in interfacial charge relaxation is instantaneous: disturbances either grow or decay in an exponential fashion, oscillatory motions are absent, and sufficiently strong fields can stabilize the jet. Next, when charge relaxation is absent oscillatory motions are again suppressed but here the jet cannot be stabilized by increasing the field strength and long wavelength disturbances are always unstable. This can be seen by noting that $F(\alpha)f_2(\alpha)$ is $O(\alpha)$ as $\alpha \to 0$.

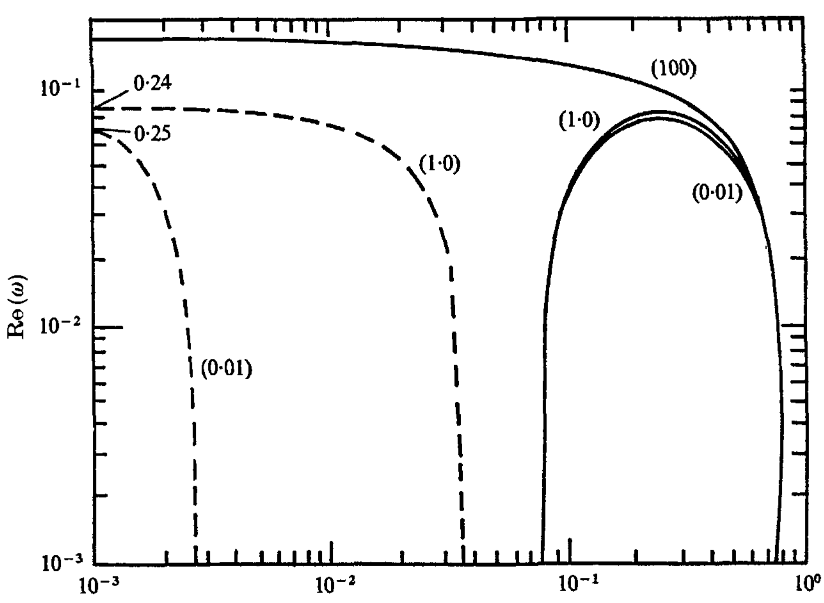

FIGURE 3. The real part of the growth rate parameter for $R \to 0$ as a function of α and the relaxation time τ_r; $E = 0{\cdot}0025$ and $\hat{K}/K = 78$. The results are for the axisymmetric mode and the dashed lines indicate an oscillatory growth. Numbers in parentheses denote values of $\overline{\tau}_r$ and the other numbers are $\mathrm{Im}\,(\omega)$.

The characteristics of the cylinder when $0 < \overline{\tau}_r < \infty$ with viscous effects dominant are different from those where inertial effects predominate in that only axisymmetric motions can be unstable. However, the addition of charge relaxation may still produce an oscillatory form of instability. Some representative results are shown on figure 3, and, it should be noted, similar results were found for smaller values of $\hat{K}/K$. In figure 3 the field strength parameter E is below that required for complete stability when charge relaxation is instantaneous. For small relaxation times there are two maxima; at small wave-numbers the instability is oscillatory while at larger wave-numbers the instability occurs without oscillation. When E is somewhat larger, e.g. $E = 0{\cdot}005$, then only the oscillatory form of instability is found when τ_r is small.

5. Summary

The stability of a fluid jet moving in an electric field aligned with the axis of the jet was studied using a theory which takes into account electrical conduction in the liquid and the relaxation of electrical charges at the interface. Subsequent to the derivation of a general dispersion relation two special situations were examined in detail: the stability of systems wherein either inertial effects or viscous effects are dominant.

It was found that whenever charge relaxation occurs disturbances may grow in an oscillatory manner, in direct contrast to the purely exponential growth characteristic of either perfect dielectrics or perfect conductors. If the fluid is a conductor, however poor, then viscous stresses must be taken into account at the interface so as to balance the electrical shearing stresses. These viscous stresses are concentrated within an electrohydrodynamic boundary layer near the interface if the viscosity is low and, due to charge relaxation phenomena, the interaction between the electrical and viscous stresses at the interface causes instability with both asymmetric and axisymmetric deformations. Furthermore, the oscillatory instability persists, no matter how large the field strength, if the relaxation processes take place at finite rates. On the other hand, the behaviour of a very viscous jet is such that even with charge relaxation only axisymmetric deformations are unstable, although the form of the instability may be oscillatory.

REFERENCES

CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press.

HUEBNER, A. L. 1969 Disintegration of charged liquid jets. *J. Fluid Mech.* **38**, 679.

LANDAU, L. D. & LIFSHITZ, E. M. 1960 *Electrodynamics of Continuous Media*. Prentice-Hall.

MELCHER, J. R. & SCHWARZ, W. J. 1968 Interfacial relaxation overstability in a tangential electric field. *Phys. Fluids*, **11**, 2604.

MELCHER, J. R. & TAYLOR, G. I. 1969 Electrohydrodynamics. In *Annual Review of Fluid Mechanics*. (Eds. W. R. Sears and M. D. Van Dyke.) Palo Alto: Annual Reviews.

NAYYAR, N. K. & MURTY, G. S. 1960 Stability of a dielectric liquid jet in the presence of a longitudinal electric field. *Proc. Phys. Soc. Lond.* **75**, 369.

SAVILLE, D. A. 1970 Electrohydrodynamic stability: fluid cylinders in longitudinal electric fields. *Phys. Fluids*, **13**, 2987.

SOMMERFELD, A. 1964 *Electrodynamics*. Academic.

TAYLOR, G. I. 1969 Electrically driven jets. *Proc. Roy. Soc.* A **313**, 453.

WATSON, G. N. 1966 *A Treatise on the Theory of Bessel Functions*. Cambridge University Press.